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The dispersion of contaminant released from instantaneous sources in laminar flow near stagnation points

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This paper considers the dispersion of a cloud of passive contaminant released from an instantaneous source in the steady two-dimensional laminar flow near the forward stagnation point on a bluff body. The body is replaced by its tangent plane $y = 0$ with x measuring distance along the plane. Far away from $y = 0$ the flow is irrotational with velocity potential $\frac{1}{2}l(x^2-y^2)$, where l is a positive constant. When the boundary layer is ignored the equation governing the distribution of concentration can be solved exactly. Consequences of this solution are that for large times the centre of mass moves parallel to the body at a speed proportional to $\exp(lt)$ while the cloud spreads out along the body symmetrically about the centre of mass with the magnitude of the spread also proportional to $\exp(lt)$. However, this solution is unrealistic because most of the contaminant is confined to a layer adjoining the body of thickness of order $(\kappa/l)^{\frac{1}{2}}$, where κ is the molecular diffusivity, and this layer normally lies within the boundary layer, which is of thickness of order $(\nu/l)^{\frac{1}{2}}$, where ν is the kinematic viscosity. An approximate analysis, based on ideas similar to those supporting the Pohlhausen method in boundary-layer theory, suggests that when the boundary layer is taken into account the conclusions above remain true provided that $\exp(lt)$ is replaced by $\exp(\beta lt)$, where β is a constant depending on ν/κ. Calculations give values of β ranging from 0·73 when $\nu/\kappa = 0{\cdot}5$ to 0·10 when $\nu/\kappa = 10^3$.

1. Introduction

The dispersion of contaminant in fluid flows is an important scientific problem, currently receiving much attention on television and in the Press. About twenty years ago Taylor (1953, 1954) discussed the way in which a finite cloud of passive contaminant is dispersed in steady pipe flow under the influence of advection with the fluid and diffusion. He showed that the centre of mass of the cloud travels downstream at the discharge velocity U and that if C_m denotes the mean of the distribution of concentration C over the pipe cross-section then for large t

$$C_m \approx \frac{1}{2(\pi Kt)^{\frac{1}{2}}} \exp\left\{-\frac{(x-Ut)^2}{4Kt}\right\}, \tag{1.1}$$

where x is axial distance and K is a constant depending on the variation of the fluid velocity over the cross-section and on the intensity of the diffusion process.

Noteworthy features of this distribution are its symmetry about the centre of mass (caused by the action of lateral diffusion on the highly asymmetrical distribution which would arise from advection alone), and that the spread of the cloud about its centre of mass is proportional to $(Kt)^{\frac{1}{2}}$.

Since Taylor's work it has been shown by Batchelor (1966) that a necessary condition for C_m to be of the form (1.1) in any flow is that the longitudinal velocity of a fluid molecule be a stationary random function of time for large t, as it is in flow in a long pipe of constant cross-section under a steady pressure gradient. In most flows this condition is not satisfied, and some of these are both important and relatively simple. For example, the dispersion of tracers in the bloodstream takes place in an unsteady flow, and in the atmosphere the time necessary for the velocity of a fluid molecule to become effectively stationary in a steady wind is so long that for practical purposes the atmosphere can often be regarded as vertically unbounded, so that the velocity of a molecule is never stationary (see Chatwin 1968).

Another important and interesting class of flows in which Batchelor's condition is not satisfied are those (statistically) steady flows in which the streamlines of the mean motion are not parallel. One of these is the subject of this paper. The common feature of these flows from the point of view of dispersion is that as a fluid molecule is advected downstream the length scale of the random motion to which it is subjected, whether molecular or turbulent, is changing with time. In certain of these flows predictions about the way in which the cloud disperses can be made on the basis of dimensional arguments (or, alternatively, by a transformation of the length and time scales so that the transformed velocity is a stationary random function of transformed time; see Batchelor (1957)). In an axisymmetric turbulent jet, for example, the statistical properties of the motion of a fluid molecule at time t after release from the origin of the jet can depend only on t and F/ρ, where F is the force applied at the origin of the jet and ρ is the fluid density. It follows that, if the longitudinal displacement of the fluid molecule at time t is $X(t)$, then $\overline{X}$, $\{\overline{(X-\overline{X})^2}\}^{\frac{1}{2}}$ and all other length scales associated with the distribution of $X(t)$ are proportional to $(Ft^2/\rho)^{\frac{1}{4}}$, where an overbar denotes the ensemble mean. It also follows that if (r, θ, x) are cylindrical polar co-ordinates centred at the origin of the jet with $r = 0$ being the axis of symmetry of the jet then $C(x, r, t)$, the distribution of concentration within a dispersing cloud, satisfies

$$C(x,r,t) = (\rho/Ft^2)^{\frac{3}{4}} f(\xi,\eta), \quad \text{where} \quad (\xi,\eta) = (\rho/Ft^2)^{\frac{1}{4}}(x,y). \tag{1.2}$$

Thus the shape of the cloud is the same for all t and its size alone changes. Unfortunately in these and similar flows it generally seems to be difficult to obtain more detailed information (about the function f in (1.2) for example) without involved computation for each flow, based on differential equations which are necessarily approximate for cases when the flow is turbulent. Such work is in progress and will be reported later.

The situation is in some respect similar to that in the early years of boundary-layer theory before the widespread use of large computers. Apart from some important similarity solutions, detailed and accurate results were difficult to obtain and, especially in engineering, approximate methods like that due to

Pohlhausen proved very useful for obtaining reasonable numerical predictions.

In the present paper such an approximate method is suggested and applied to the dispersion of a cloud of contaminant in the laminar flow near a stagnation point. Although it is believed that the method will be useful in other flows to which Taylor's analysis cannot be applied and which were briefly discussed above, this particular flow was chosen because of the lucky circumstance that the diffusion equation governing the dispersion of the cloud can be solved exactly when the contaminant is in the irrotational part of the flow outside the boundary layer. The form of this solution (see § 2) suggested the approximate method proposed and discussed in §§ 3 and 4 and also enables some checking of its predictions. These results illustrate the effect of the spreading of the mean streamlines. This effect was also discussed for *turbulent* flow and *steady* sources near stagnation points by Hunt & Mulhearn (1973) and, although the differences between their problem and the one examined here make precise comparison impossible, some of the qualitative features are similar and will be mentioned later.

In the neighbourhood of the forward stagnation point O on any bluff body the flow is approximately that which would occur were the body replaced by its tangent plane at O, say $y = 0$, with x and z being co-ordinates in the plane. Outside the boundary layer the flow is irrotational and derivable from the velocity potential ϕ, where

$$\phi = \tfrac{1}{2}\{lx^2 + mz^2 - (l+m)y^2\}, \tag{1.3}$$

and l and m are constants, and $l+m$ must be positive for O to be the forward stagnation point. Hence at least one of l and m is positive and it will be supposed that $l > 0$. For simplicity of exposition the detailed calculations in this paper are for the two-dimensional case when $m = 0$, but the work can be generalized straightforwardly to three dimensions.

This restriction to two dimensions means that in later work in this paper $C(x, y, t)$ is the distribution of concentration in the x, y plane of contaminant initially distributed uniformly along a *line* parallel to the z axis. However, the same function is relevant in the case when $m = 0$ and the contaminant is initially concentrated at a *point* situated, say, in the plane $z = 0$, for dispersion in the direction of the z axis is caused by molecular diffusion alone, so that the distribution of concentration throughout space is given simply by

$$C(x, y, t)\frac{\exp\{-z^2/4\kappa t\}}{2(\pi\kappa t)^{\frac{1}{2}}}, \tag{1.4}$$

where κ is the molecular diffusivity. In both cases $C(x, y, t)\,\delta x\,\delta y$ is the proportion of contaminant within an infinite cylinder parallel to the z axis with cross-sectional area $\delta x\,\delta y$. This last remark applies even for contaminant initially distributed *non-uniformly* along a line parallel to the z axis.

The boundary layer which forms on $y = 0$ is discussed in Rosenhead (1963). It has the property that its thickness is constant and thus is proportional to $(\nu/l)^{\frac{1}{2}}$ on dimensional grounds. When $m = 0$ the stream function $\Psi(x, y)$, therefore, has the form

$$\Psi(x, y) = x(\nu l)^{\frac{1}{2}} F(\eta), \quad \eta = y(l/\nu)^{\frac{1}{2}}. \tag{1.5}$$

Use of the boundary-layer equations (or indeed the Navier–Stokes equations) and the appropriate boundary conditions leads to

$$\left.\begin{aligned} F''' + FF'' + 1 - F'^2 = 0, \\ F(0) = F'(0) = 0, \quad F'(\infty) = 1. \end{aligned}\right\} \tag{1.6}$$

This equation is due to Hiemenz although the values of F and its derivatives used in this paper were calculated by Bickley (unpublished, but quoted in Rosenhead 1963, p. 232). Note that (1.5) may also be used to describe the irrotational flow, with $m = 0$, when

$$F(\eta) = \eta. \tag{1.7}$$

In the remainder of this paper it will prove convenient to use $F(\eta)$ to describe either the boundary-layer flow [with (1.6)] or the irrotational flow [with (1.7)], and the context will make clear which form is being used.

2. Dispersion in the irrotational flow outside the boundary layer

When $m = 0$ the irrotational flow has components $(lx, -ly, 0)$, so that the equation governing the distribution of concentration $C(x, y, t)$ is

$$\frac{\partial C}{\partial t} + lx\frac{\partial C}{\partial x} - ly\frac{\partial C}{\partial y} = \kappa\frac{\partial^2 C}{\partial x^2} + \kappa\frac{\partial^2 C}{\partial y^2}, \tag{2.1}$$

where κ is the molecular diffusivity, here assumed constant. Although κ depends on C when C is small previous work has given good agreement between theory and experiment when κ is assumed constant in the case of laminar flow in pipes of constant cross-section (see for example Taylor 1953). It is assumed likewise here that κ is constant since the purpose of the present paper is to examine the effect of non-parallel streamlines rather than that of variable κ. If the contaminant is released from a source at (x_0, y_0) at $t = 0$ then

$$C(x, y, 0) = \delta(x - x_0)\,\delta(y - y_0). \tag{2.2}$$

Also there must be no flux of contaminant across the boundary and C must decay to zero far away from the source. Hence

$$\partial C/\partial y = 0 \quad \text{at} \quad y = 0, \qquad C \to 0 \quad \text{as} \quad x^2 + y^2 \to \infty. \tag{2.3}$$

The solution of this problem is

$$C = \frac{l}{4\pi\kappa \sinh lt}\exp\left\{-\frac{l(x - x_0 e^{lt})^2}{2\kappa(e^{2lt} - 1)}\right\}\left[\exp\left\{-\frac{l(y - y_0 e^{-lt})^2}{2\kappa(1 - e^{-2lt})}\right\} + \exp\left\{-\frac{l(y + y_0 e^{-lt})^2}{2\kappa(1 - e^{-2lt})}\right\}\right]. \tag{2.4}$$

The particular form of (2.4) with $x_0 = y_0 = 0$ was derived by Townsend (1951), but in the context of the diffusion of a heat spot in turbulent flow. As might be expected the solution for $y_0 \neq 0$ displays the effect of an image source at $(x_0, -y_0)$. The form of $C(x, y, t)$, given in (2.4), is illustrated for various values of lt and (x_0, y_0) in figure 1. Certain features are worth pointing out for later use.

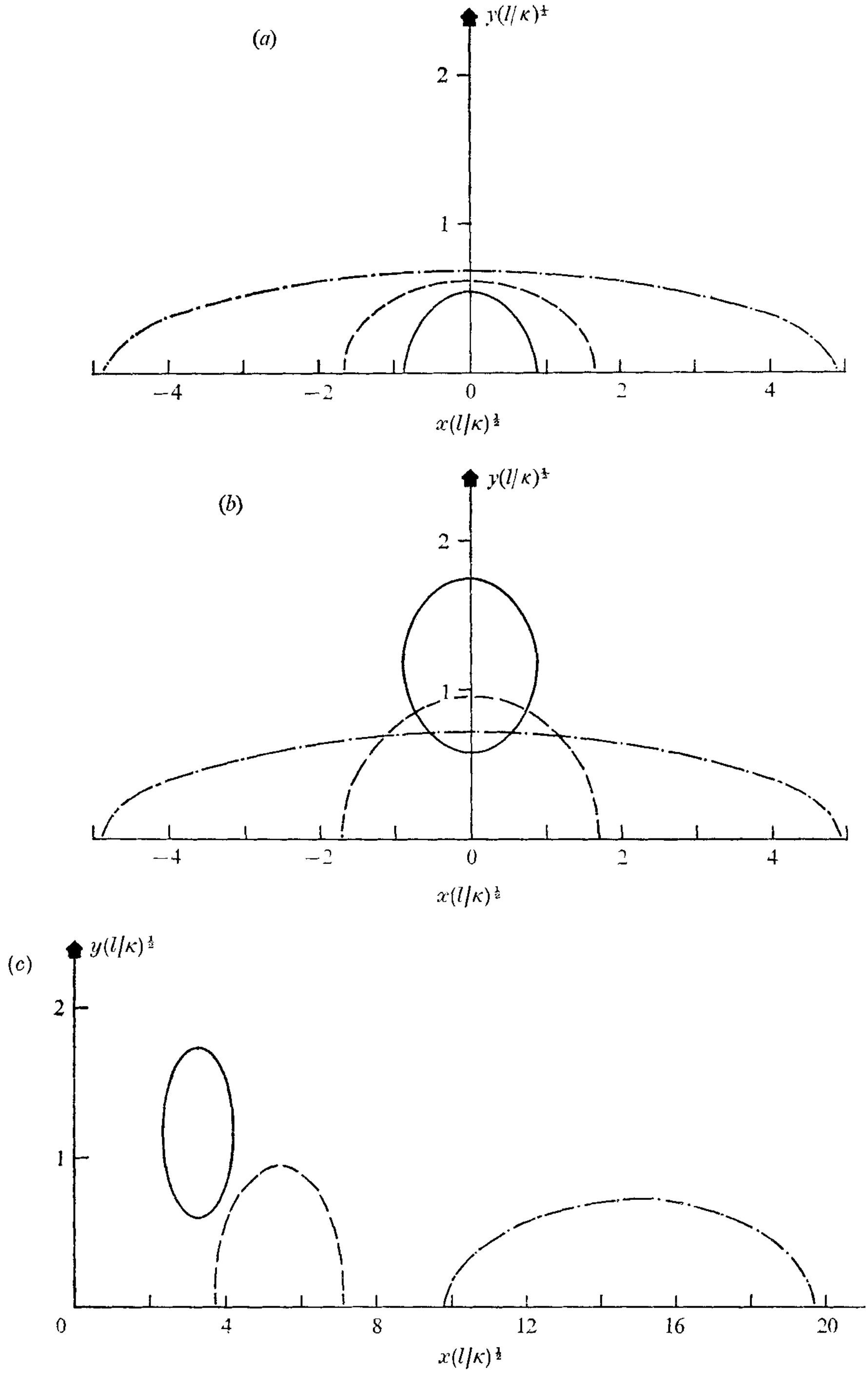

FIGURE 1. Contours of C in irrotational flow obtained from (2.4) for various values of x_0, y_0 and lt. Each contour has C equal to 0·8 of the maximum concentration at that value of lt. (*a*) $x_0 = 0$, $y_0 = 0$; (*b*) $x_0 = 0$, $y_0(l/\kappa)^{\frac{1}{2}} = 2$; (*c*) $x_0(l/\kappa)^{\frac{1}{2}} = 2$, $y_0(l/\kappa)^{\frac{1}{2}} = 2$. ——, $lt = \frac{1}{2}$; – – –, $lt = 1$; —·—·, $lt = 2$.

The centre of mass of the cloud of contaminant can be obtained by integrating (2.4). Its x co-ordinate $\bar{X}(t)$ satisfies

$$\bar{X}(t) = x_0 e^{lt}, \tag{2.5}$$

and this is *precisely* the x co-ordinate at time t of the fluid particle that was at the source position (x_0, y_0) at $t = 0$. Thus, at least in the x direction, the centre of mass of the cloud moves with the fluid. In a sense this is similar to the situation in pipe flow where the centre of mass of the cloud moves with the discharge velocity [see (1.1)], which is the average fluid velocity (over the cross-section). However, in pipe flow it is diffusion *across* the streamlines that causes the molecules of contaminant to sample all fluid velocities, giving an average contaminant molecule velocity equal to the average fluid velocity, whereas in stagnation-point flow different contaminant molecules have different velocities in the x direction because of diffusion *along* the direction of flow. But (2.5) shows that those parts of the cloud that are moving more slowly than the centre of mass give a contribution to the position of the centre of mass that is exactly balanced by those parts that are moving more quickly.

Even more surprising perhaps, though clearly shown in figure 1, is the fact that the cloud is spread out in the x direction symmetrically about $\bar{X}(t)$. It can be verified that this result is, unlike the corresponding result in pipe flow, independent of lateral diffusion (that in the y direction) and evidently depends on the fact that the x component of fluid velocity relative to that of the centre of mass is an odd function of displacement from the centre of mass. The magnitude of the spread can be measured by $\Sigma_x(t)$, where

$$\Sigma_x^2(t) = \iint (x-\bar{X})^2\, C\, dx\, dy. \tag{2.6}$$

From (2.4) the following value is obtained:

$$\Sigma_x = \{\kappa[\exp(2lt) - 1]/l\}^{\frac{1}{2}}. \tag{2.7}$$

For small lt this is approximately $(2\kappa t)^{\frac{1}{2}}$, which is the spread which would be observed with no advection, but for large lt the value of Σ_x increases like $\exp(lt)$ as does the displacement in the x direction between any pair of fluid molecules.

The y component of velocity is everywhere towards $y = 0$, so that diffusion cannot spread the cloud of contaminant over all values of y. Indeed (2.4) shows, and figure 1 illustrates, that for large values of lt the contaminant is confined to a layer of constant and uniform thickness of order $(\kappa/l)^{\frac{1}{2}}$, and that, again for large values of lt,

$$\bar{Y} \approx (2\kappa/\pi l)^{\frac{1}{2}}, \quad \Sigma_y \approx \{\kappa(1-2/\pi)/l\}^{\frac{1}{2}}, \tag{2.8}$$

where $\bar{Y}$ is the y co-ordinate of the centre of mass of the cloud and Σ_y is its spread in the y direction defined by a relation analogous to (2.6). As is to be anticipated the presence of the wall ensures that both $\bar{Y}$ and Σ_y are eventually independent of y_0. Indeed more detailed calculations show that, for all values of lt, Σ_y (like Σ_x) is independent of (x_0, y_0) and that $\bar{Y}$ depends on y_0 only for small values of lt, when, like $\bar{X}$, it follows the fluid particle which was initially at (x_0, y_0).

Hunt & Mulhearn (1973) found that the ensemble mean position of a marked fluid particle in weak turbulent flow tended to follow the mean streamline of the flow through the initial position of the particle at least until the influence of the boundary became important. The results above for $\bar{X}$ and $\bar{Y}$ for small lt show the same result for a marked fluid molecule in laminar flow. They also found that for turbulent flow near the forward stagnation point on a circular cylinder the cloud of contaminant released from a source on $x = 0$ (using the notation of the present paper) tended to flatten along the cylinder as time progressed. Figures 1 (*a*) and (*b*) show the same phenomenon in the present problem.

Unfortunately this exact solution has little practical significance since in most fluids $\nu/\kappa > 1$ and therefore the layer of contaminant (of thickness of order $(\kappa/l)^{\frac{1}{2}}$) does not lie mainly outside the boundary layer (of thickness of order $(\nu/l)^{\frac{1}{2}}$; see (1.5)). In liquids indeed, the value of ν/κ is of order 10^3, so that the layer of contaminant lies deep within the boundary layer for all lt if $y_0 = 0$, and eventually if $y_0 \neq 0$.

In the latter case, i.e. $y_0 \neq 0$, the solution (2.4) may have significance for lt not too large. This will happen provided that there is a range of values of lt for which a substantial portion of the cloud lies outside the boundary layer but within the region where the velocity has the form derived from (1.3) with $m = 0$. Consider for example flow of a uniform stream of velocity U at infinity past a circular cylinder of radius a in which l is $2U/a$. Then for the above conditions to be satisfied initially it is necessary that

$$a^{-1}(\nu/l)^{\frac{1}{2}} = (\nu/2Ua)^{\frac{1}{2}} \ll y_0/a \ll 1, \quad x_0/a \ll 1. \tag{2.9}$$

Subsequently the solution (2.4) has validity for those times for which (2.9) hold with x_0 and y_0 replaced by $\bar{X}$ and $\bar{Y}$, or, if x_0 (and thus $\bar{X}$) is zero, with x_0 replaced by Σ_x. Use of (2.4), (2.5) and (2.7) gives validity for those times for which

$$\left.\begin{aligned} &\exp(2Ut/a) \ll (y_0/a)(2Ua/\nu)^{\frac{1}{2}}, \\ \exp(2Ut/a) \ll a/x_0 \quad (x_0 \neq 0) \quad &\text{or} \quad \exp(2Ut/a) \ll (2Ua/\kappa)^{\frac{1}{2}} \quad (x_0 = 0). \end{aligned}\right\} \tag{2.10}$$

For a case in which $2Ua/\nu = 10^6$ and $x_0/a = y_0/a = 10^{-1}$ the most restrictive inequality of (2.10) is that involving x_0 and implies validity provided that $Ut = O(a)$, i.e. for times in which a fluid particle far away from the cylinder travels a distance equal to the cylinder radius. If $x_0 = 0$ but the other values above remain unchanged the most restrictive inequality of (2.10) is that involving y_0 and implies validity provided that $Ut = O(10a)$.

3. Dispersion in the boundary layer

The analysis at the end of §2 shows that, whatever the values of x_0 and y_0, equation (2.1) for C is not valid for large lt, and has to be modified by using the boundary-layer velocity distribution defined by (1.5) and (1.6). Writing $\eta = y(l/\nu)^{\frac{1}{2}}$ as in (1.5) gives

$$\frac{\partial C}{\partial t} + lxF'(\eta)\frac{\partial C}{\partial x} - lF(\eta)\frac{\partial C}{\partial \eta} = \kappa\frac{\partial^2 C}{\partial x^2} + (\kappa l/\nu)\frac{\partial^2 C}{\partial \eta^2}. \tag{3.1}$$

It is convenient to use dimensionless variables throughout, using scales for x and t suggested by the form of the solution (2.4) for irrotational flow. On writing

$$\xi = x(l/\kappa)^{\frac{1}{2}}, \quad \tau = lt, \quad \sigma = \nu/\kappa, \tag{3.2}$$

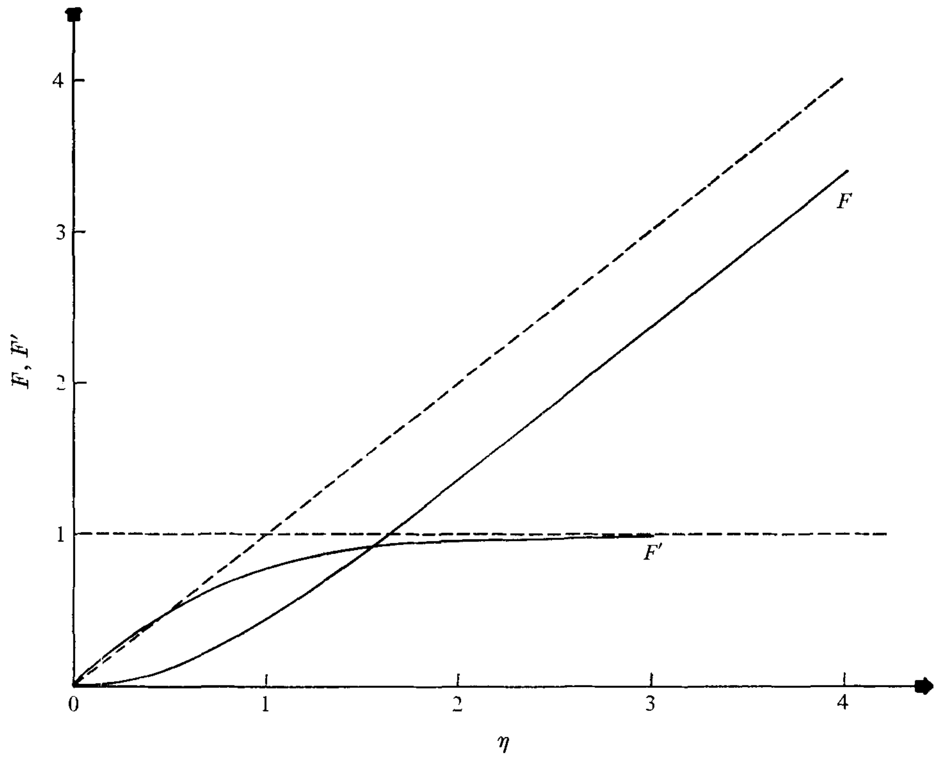

FIGURE 2. Graphs of F and F' defined in (1.5) and (1.6), shown by the solid lines. These have been drawn from data on p. 232 of Rosenhead (1963). The dashed lines are the corresponding curves in irrotational flow, in which $F(\eta) = \eta$ [see (1.7)].

(3.1) becomes

$$\frac{\partial C}{\partial \tau}+\xi F'\frac{\partial C}{\partial \xi}-F\frac{\partial C}{\partial \eta}=\frac{\partial^2 C}{\partial \xi^2}+\frac{1}{\sigma}\frac{\partial^2 C}{\partial \eta^2}. \tag{3.3}$$

The boundary conditions for C are the same as those in irrotational flow, viz. (2.2) and (2.3).

Figure 2 shows how F and F' in boundary-layer flow [given in (1.5) and (1.6)] compare with their forms in irrotational flow [given in (1.5) and (1.7)]. The boundary-layer thickness, defined as the distance from the wall at which the x component of velocity is 99 % of its value in the main stream, is $2{\cdot}4(\nu/l)^{\frac{1}{2}}$, and for large η, $F \approx \eta - 0{\cdot}68$ in the boundary layer.

The component of velocity towards the wall is less than in irrotational flow, so that the thickness of the layer of contaminant will be greater than in irrotational flow. For given x, the x component of velocity is less than its value in irrotational flow, so that the values of $\overline{X}$ and Σ_x will be less than in irrotational flow. The form of (3.3) shows that the magnitudes of these differences depend on the value of σ, and it is one purpose of the remainder of this paper to quantify these differences at least approximately.

With F having its boundary-layer form given in (1.5) and (1.6) there is apparently no simple analytic form for the solution of (3.3), but the work of Aris

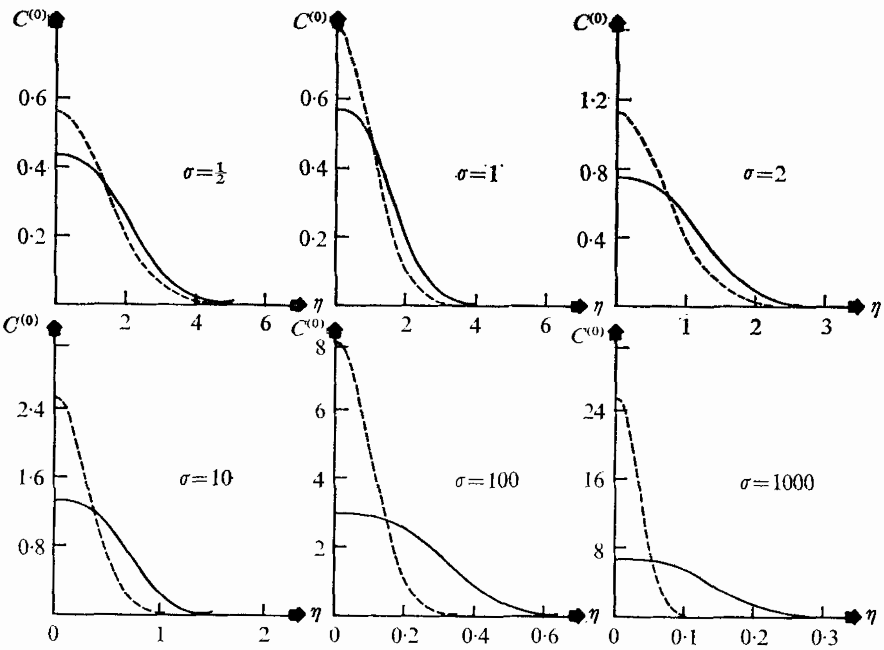

FIGURE 3. Graphs of $C^{(0)}$ in boundary-layer flow defined in (3.6), for various values of σ, shown by the solid lines. The dashed lines are the corresponding curves in irrotational flow. Notice that the scales are not the same for each graph.

(1956) has shown that it is often useful and analytically practicable to look at the form of the integral moments of the distribution of concentration. Here the integral moments $C^{(n)}(\eta, \tau)$ in the x direction, defined for all positive integers n by

$$C^{(n)}(\eta,\tau) = \int_{-\infty}^{\infty} \xi^n C(\xi,\eta,\tau)\, d\xi \quad (n = 0, 1, 2, \ldots), \tag{3.4}$$

give information about how the distribution of concentration varies with η. For example $C^{(0)}\delta\eta$ is proportional to the total amount of contaminant between η and $\eta+\delta\eta$ and an equation for $C^{(0)}$ can be obtained by integrating (3.3) with respect to ξ, giving

$$\frac{\partial C^{(0)}}{\partial \tau} - \frac{\partial}{\partial \eta}\{FC^{(0)}\} - \frac{1}{\sigma}\frac{\partial^2 C^{(0)}}{\partial \eta^2} = 0. \tag{3.5}$$

It is evident that, as $\tau \to \infty$, $C^{(0)}$ becomes independent of τ and so, integrating (3.5),

$$C^{(0)} \approx A \exp\left\{-\sigma \int_0^{\eta} F(u)\, du\right\}, \tag{3.6}$$

where A is a constant. In figure 3 the large time value of $C^{(0)}$ given by (3.6) is plotted for various values of σ and compared with the corresponding result in irrotational flow. As anticipated above the contaminant layer is thicker than is predicted by irrotational flow theory.

The values of $C^{(n)}$ for values of n greater than zero also have relevance. For example $C^{(1)}$, $C^{(2)}$ and $C^{(3)}$ are respectively related to the centre of mass, the variance and the skewness of the distribution of concentration at a given height. An equation for $C^{(n)}$ can be obtained by multiplying (3.3) by ξ^n and integrating. This gives

$$\frac{\partial C^{(n)}}{\partial \tau} - nF'C^{(n)} - \frac{\partial}{\partial \eta}\{FC^{(n)}\} - \frac{1}{\sigma}\frac{\partial^2 C^{(n)}}{\partial \eta^2} = n(n-1)\,C^{(n-2)}, \tag{3.7}$$

for all n provided that the right-hand side of (3.7) is taken as zero for $n = 0$ and $n = 1$. (Thus (3.7) reduces to (3.5) when $n = 0$.) To discuss this equation further it is convenient to define

$$\mathscr{C}^{(n)}(\eta, \tau) \equiv C^{(n)}(\eta, \tau) \exp\left\{\tfrac{1}{2}\sigma \int_0^\eta F(u)\,du\right\}, \tag{3.8}$$

so that

$$\frac{\partial^2 \mathscr{C}^{(n)}}{\partial \eta^2} + \left[-\sigma\frac{\partial \mathscr{C}^{(n)}}{\partial \tau} - \{\tfrac{1}{4}\sigma^2F^2 - \sigma(n+\tfrac{1}{2})\,F'\}\,\mathscr{C}^{(n)}\right] = -n(n-1)\,\sigma\mathscr{C}^{(n-2)}. \tag{3.9}$$

The complementary function of (3.9) can be written as

$$\mathscr{C}^{(n)} = \sum_{m=0}^{\infty} A_m^{(n)} f_m^{(n)}(\eta) \exp(-\alpha_m^{(n)}\tau), \tag{3.10}$$

where $f_m^{(n)}$ is defined between $\eta = 0$ and $\eta = \infty$ for all integers m and n by

$$d^2 f_m^{(n)}/d\eta^2 + [\sigma\alpha_m^{(n)} - \{\tfrac{1}{4}\sigma^2F^2 - \sigma(n+\tfrac{1}{2})\,F'\}]\,f_m^{(n)} = 0. \tag{3.11}$$

The boundary conditions on $f_m^{(n)}$ can be obtained from (2.3) as

$$df_m^{(n)}/d\eta = 0 \quad \text{at} \quad \eta = 0, \quad f_m^{(n)} \to 0 \quad \text{as} \quad \eta \to \infty. \tag{3.12}$$

For each n, (3.11) and (3.12) constitute a standard eigenvalue problem so that $\alpha_0^{(n)} < \alpha_1^{(n)} < \alpha_2^{(n)} \ldots$. The work following (3.5) shows that $\alpha_0^{(0)} = 0$ and it can also be shown (see Titchmarsh 1962, chap. 5) that $\alpha_0^{(p)} < \alpha_0^{(q)} < \alpha_0^{(0)} = 0$ when $p > q > 0$. Since it follows from (3.10) that, for large τ, $\mathscr{C}^{(n)}$, and hence $C^{(n)}$, is proportional to $\exp(-\alpha_0^{(n)}\tau)$, this shows that, for $n > 0$, $C^{(n)}$ increases exponentially with τ. In particular, for large τ, $\bar{X}$ and Σ_x are proportional to $\exp(-\alpha_0^{(1)}\tau)$ and $\exp(-\tfrac{1}{2}\alpha_0^{(2)}\tau)$ respectively. A calculation by hand gave

$$\alpha_0^{(1)} \approx -0{\cdot}70, \quad \alpha_0^{(2)} \approx -1{\cdot}46 \quad \text{for} \quad \sigma = 1, \tag{3.13}$$

while in irrotational flow the values given by (2.5) and (2.7) are $\alpha_0^{(1)} = -1$ and $\alpha_0^{(2)} = -2$. These results support the conclusion above that both $\bar{X}$ and Σ_x are less than their values in irrotational flow.

A thorough numerical investigation of the equation for C itself, viz. (3.3), or of those for the $C^{(n)}$, viz. (3.7), for a range of values of σ is intended, but is unlikely to lead to great insight into the way in which contaminant disperses in that whole class of flows discussed in §1, of which two-dimensional stagnation-point flow is but one example.

4. An approximate method for large times

In irrotational flow the layer of contaminant adjacent to $y = 0$ is of constant and uniform thickness as $\tau \to \infty$ and (2.4) shows that, for large τ,

$$C \approx A(\xi, \tau) \exp\{-\tfrac{1}{2}\sigma\eta^2\}. \tag{4.1}$$

Hence at each η the variation of C with ξ and τ is the same. In the boundary layer the stream function is proportional to x as it is in irrotational flow and hence the boundary layer is of uniform thickness. It therefore seems plausible that the layer of contaminant in the boundary layer will also be of uniform thickness as $\tau \to \infty$. Now in irrotational flow the variation of $C^{(0)}$ with η can be obtained from (3.6) with F given by (1.7) and is the same as the variation of C with η given by (4.1). Accordingly the hypothesis now made is that this result is approximately true in the boundary-layer flow for large τ, i.e. that

$$C \approx A(\xi, \tau) \exp\left\{-\sigma \int_0^\eta F(u)\,du\right\}, \tag{4.2}$$

using (3.6). Substitution of (4.2) into the *exact* equation for C, equation (3.3), gives

$$\frac{\partial A}{\partial \tau} + F' \frac{\partial}{\partial \xi}(\xi A) = \frac{\partial^2 A}{\partial \xi^2}, \tag{4.3}$$

which *cannot* be true unless F' is independent of η. Thus (4.3) is exact in irrotational flow, when (1.7) holds, but not in the boundary-layer flow, when (1.6) holds.

However, F' describes the variation of the x component of velocity normal to the body and it seems reasonable to suppose that first approximations to $\overline{X}$, Σ_x and other properties of the dispersing cloud can be obtained by using an average value of F' in (4.3). Many averages could be proposed but one only is consistent with the requirement that contaminant be conserved, and also with (4.2). Integrate (3.3) with respect to η from 0 to ∞ obtaining

$$\frac{\partial}{\partial \tau}\left\{\int_0^\infty C\,d\eta\right\} + \frac{\partial}{\partial \xi}\left\{\xi \int_0^\infty F'C\,d\eta\right\} = \frac{\partial^2}{\partial \xi^2}\left\{\int_0^\infty C\,d\eta\right\}. \tag{4.4}$$

Since

$$\delta\xi \int_0^\infty C\,d\eta$$

is, in non-dimensional variables, the quantity of contaminant between two infinite planes perpendicular to the body a small distance $\delta\xi$ apart, (4.4) is the equation describing how this quantity changes owing to advection and diffusion. Equation (4.4) is obtained from (3.3) in the same way as, in boundary-layer theory, the momentum integral of von Kármán is obtained from the boundary-layer equations.

Now substitution of the assumed form of C, given in (4.2), into (4.4) leads to an equation for $A(\xi, \tau)$ and, again, this procedure is similar to that employed in those methods, like that of Pohlhausen, which use the momentum integral of von Kármán to obtain approximate predictions of the behaviour of boundary layers. The equation obtained for $A(\xi, \tau)$ in this way is

$$\frac{\partial A}{\partial \tau} + \beta \frac{\partial}{\partial \xi}(\xi A) = \frac{\partial^2 A}{\partial \xi^2}, \tag{4.5}$$

σ	0·5	1·0	2·0	5·0	10·0	100·0	1000·0	∞
β	0·73	0·66	0·58	0·48	0·40	0·21	0·10	0

TABLE 1. The values of β, given by (4.6), in terms of $\sigma = \nu/\kappa$. The data used in calculating β is given on page 232 of Rosenhead (1963). In irrotational flow $\beta = 1$.

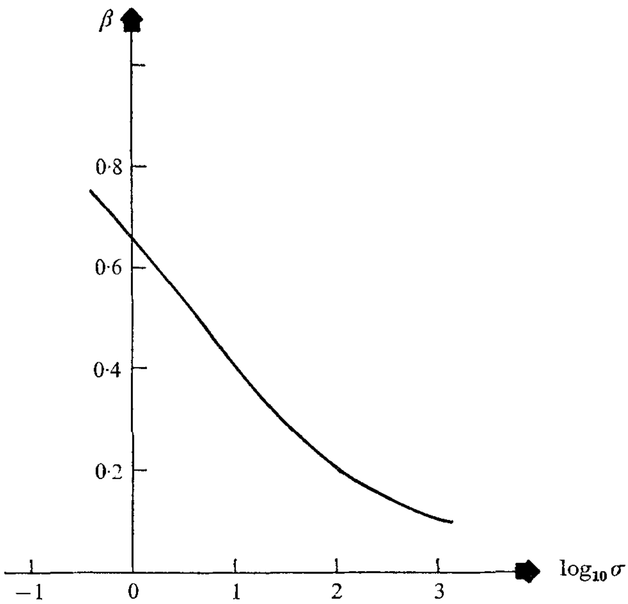

FIGURE 4. Graph of β against $\log_{10}\sigma$ (see table 1). As σ becomes very large β tends to zero and in irrotational flow β is 1.

where

$$\beta = \int_0^\infty F'(\eta)\exp\left\{-\sigma\int_0^\eta F(u)\,du\right\}d\eta \Big/ \int_0^\infty \exp\left\{-\sigma\int_0^\eta F(u)\,du\right\}d\eta. \quad (4.6)$$

Comparison of (4.3) and (4.5) shows that F' has been replaced by an average value β, and (4.6) shows that this average of F' is obtained by weighting F' with the postulated variation of C normal to the wall.

The solution of (4.5) which tends to zero as $|\xi| \to \infty$ is, returning to dimensional variables,

$$A(x,t) \propto \left\{\frac{l}{\kappa(e^{2\beta lt}-1)}\right\}^{\frac{1}{2}} \exp\left\{-\frac{\beta l(x-x_0e^{\beta lt})^2}{2\kappa(e^{2\beta lt}-1)}\right\}, \quad (4.7)$$

where x_0 is a constant (which need not be the x co-ordinate of the actual source since (4.7) has been obtained on the assumption that $lt \gg 1$). From (2.4) it can be seen that (4.7) is correct in irrotational flow provided $\beta = 1$. But (4.6) shows that this is true, for in irrotational flow $F'(\eta) = 1$ everywhere [see (1.7)].

For $lt \gg 1$, according to (4.7),

$$\bar{X} \approx x_0 \exp(\beta lt), \quad \Sigma_x \approx \{(\kappa/\beta l)[\exp(2\beta lt)-1]\}^{\frac{1}{2}}, \quad (4.8)$$

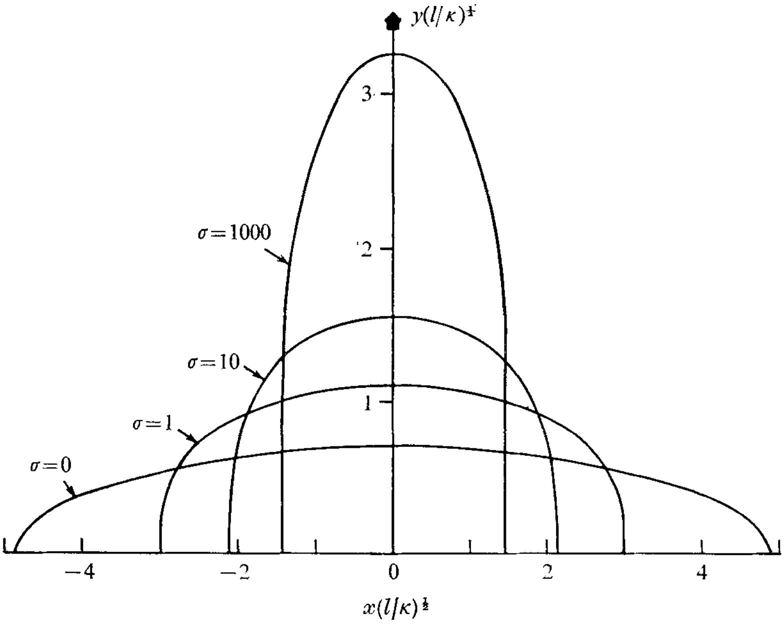

FIGURE 5. Contours of C given by (4.2) and (4.7) with $x_0 = 0$ and $lt = 2$ for various values of σ. As in figure 1 each contour has C equal to 0·8 of the maximum concentration at $lt = 2$. At this value of lt the value of y_0 has little significance provided that it is of order $(\kappa/l)^{\frac{1}{2}}$. The irrotational flow contour is shown for comparison, and is marked $\sigma = 0$.

so that β determines both the speed of the centre of mass of the cloud and the rate at which it spreads in the x direction. Values of β for various values of σ are given in table 1 and plotted in figure 4. The graph shows that, for $\frac{1}{2} < \sigma < 10$,

$$\beta \approx 0{\cdot}66 - 0{\cdot}26 \log_{10} \sigma, \tag{4.9}$$

but that, for higher values of σ, β decreases less rapidly than predicted by (4.9). Since $\sigma = \nu/\kappa$ measures the relative intensity of viscous and molecular diffusion it is evident that for large values of σ the contaminant layer is embedded deep within the boundary layer (as shown in figure 3). Because the x component of velocity increases monotonically from zero at the wall the value of β decreases with σ in agreement with figure 4.

The form of C given by (4.2) with (4.7) is plotted in figure 5 for $\sigma = 1$, 10 and 1000 and $lt = 2$ and compared with the form in irrotational flow. Notice particularly how, as σ increases, the cloud spreads less in the x direction for the reasons outlined in the previous paragraph. This figure, in contrast with figure 3, has $y(l/\kappa)^{\frac{1}{2}}$ as ordinate so that comparison for various values of σ can be made.

Judgement of the method proposed here must be postponed until accurate numerical results are available, but one implication will be mentioned here. This is that the eigenvalues $\alpha_0^{(n)}$ of the moment equations [see (3.11)] satisfy

$$\alpha_0^{(n)} = n\alpha_0^{(0)} = -n\beta, \tag{4.10}$$

as they do in irrotational flow. This result follows by determining the integral moments of (4.7). For $\sigma = 1$ this gives $\alpha_0^{(1)} = -0{\cdot}66$ and $\alpha_0^{(2)} = -1{\cdot}32$, which are fairly close to the values given in (3.13).

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